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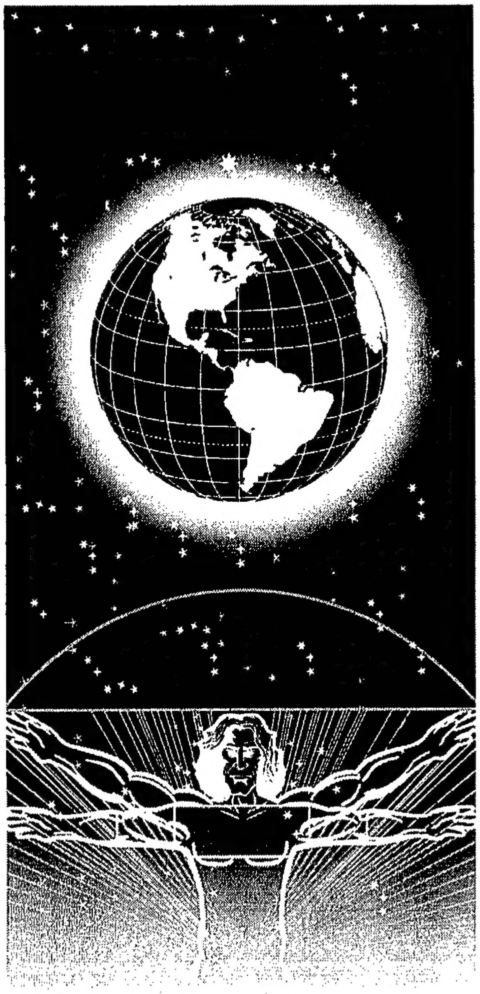

UNITED STATES AIR FORCE RESEARCH LABORATORY

NUMERICAL ANALYSIS OF A SINGULAR INTEGRAL EQUATION ARISING FROM ELECTROMAGNETIC INTERIOR SCATTERING

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14. ABSTRACT
Solving interior scattering problems of electromagnetic waves in biological media using a volume integral equation approach gives rise naturally to an integral equation with unbounded coefficients and with strongly singular kernel. The numerical solution of an analogous 1-D integral equation of the second kind is studied and a convergence proof for the numerical solution is given.

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Contents

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1 Introduction

For safety and health reasons, it is of considerable interest to assess the short- and long-term effects of electromagnetic (EM) radiation on people working near radars and other similar EM-wave-generating devices. Research to understand this can be classified as epidemiological, experimental, and numerical. In numerical electromagnetic dosimetry one is led naturally to the problem of solving the Maxwell's Equations inside a highly inhomogeneous and highly dispersive body. One of the solution approaches is to solve an equivalent problem in the frequency domain using a volume integral equation formulation.

Mathematically, in the time-harmonic case, if the body (V) is incident by an electric field $\mathbf{E}^{\mathrm{i}}(\mathbf{r})$ and if $\mathbf{E}(\mathbf{r})$ is the total electric field inside the body ($\mathbf{r} \in V$), then the scattered field, defined as the difference between the two fields, $\mathbf{E}^{\mathrm{s}}(\mathbf{r}) := \mathbf{E}(\mathbf{r}) - \mathbf{E}^{\mathrm{i}}(\mathbf{r})$, can be shown to take the form

$$\mathbf{E}^{\mathrm{s}}(\mathbf{r}) = (\mathbf{I} + \frac{1}{k_o^2}\nabla\nabla\cdot)\int_V g(\mathbf{r},\mathbf{r}')\ \mathbf{F}_{\mathbf{E}}(\mathbf{r}')\ dV' \tag{1}$$

in which

$$\begin{aligned}
\mathbf{F}_{\mathbf{E}}(\mathbf{r}) &:= \tau(\mathbf{r})\ \mathbf{E}(\mathbf{r}) \\
\tau(\mathbf{r}) &:= k^2(\mathbf{r}) - k_o^2 \\
g(\mathbf{r},\mathbf{r}') &:= \frac{e^{jk_o r}}{4\pi r} \\
r &:= |\,\mathbf{r} - \mathbf{r}'\,|
\end{aligned}$$

Here $j = \sqrt{-1}$ and k_o and $k(\mathbf{r})$ are the wave numbers associated with free space and the body respectively. Further manipulation of Equation (1) to move the differentiations under the integral signs results readily in a vector integral equation of the form (see [7] for details)

$$\mathbf{E}^{\mathrm{s}}(\mathbf{r}) = \bar{\mathbf{A}}(\mathbf{r})\,\mathbf{F}_{\mathbf{E}}(\mathbf{r}) + \int_V \bar{\mathbf{G}}(\mathbf{r},\mathbf{r}')(\mathbf{F}_{\mathbf{E}}(\mathbf{r}') - \mathbf{F}_{\mathbf{E}}(\mathbf{r}))\ dV' \tag{2}$$

in which the dyad $\bar{\mathbf{A}}(\mathbf{r})$ becomes unbounded near the boundary and the dyad (the dyadic Green's function) $\bar{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ has a singularity of type $O(r^{-3})$. Previous attempts to solve this equation [6, 7] using a Moment Method or a Nyström Method have been successful only

for a restricted class of parameters. Before attempting to conduct a thorough numerical analysis of Equation (2), we opted instead to analyze the following simpler but analogous 1-D integral equation:

$$\lambda(t)\,\phi(t) - K_g\phi(t) - K_b\phi(t) = \chi_1(t) \tag{3}$$

for $t \in [a,b]$ and $\chi_1 \in C([a,b])$. Here, analogous to Equation (2),

$$\begin{aligned} K_g\phi(t) &= \int_a^b k_g(t,s)\,\phi(s)\,ds \\ K_b\phi(t) &= \int_a^b k_b(t,s)\,[\phi(s) - \phi(t)]\,ds \end{aligned}$$

The functions $\lambda(t), k_b(t,s)$, and $k_g(t,s)$ also have properties analogous to that in the 3-D case, namely:

- $\lambda(t) > 0$ on (a,b)
- $\lim_{t\to a} \lambda(t) = \infty$ and $\lim_{t\to b} \lambda(t) = \infty$.
- $k_b(t,s) = \frac{\gamma_1}{|t-s|}$, $\gamma_1 > 0$.
- $k_g(t,s)$ is continuous on $[a,b] \times [a,b]$.

It should be mentioned that the problem considered here is not equivalent to the 1-D Maxwell's Equations wherein **E** is dependent on only one spatial dimension. It is well-known that the Green's function for the 1-D Maxwell's Equations is much better behaved. To keep the problem here simpler, however, we will ignore the 'good' kernel $k_g(t,s)$ in the following analysis and concentrate only on the equation

$$\lambda(t)\,\phi(t) - K_b\phi(t) = \chi_1(t) \tag{4}$$

As a preliminary analysis of Equation (3), we studied numerically the 1-D problem analogous to Equation (2) in which

$$g(t,s) = |t-s|\,(\ln|t-s| - 1).$$

After properly moving the double differentiations under the integral sign, the resulting $k_g(t,s)$, $k_b(t,s)$ and $\lambda(t)$ can be shown to satisfy the properties mentioned above. Numerical solutions were obtained using several variants of the Nyström methods:

1. Product method with extrapolation at the end intervals
2. Gauss-Legendre method

3. A simple method (in which uniformly spaced integration points and uniform weights are used)

In each case, apparent convergence was obtained. The main result of this study is a rigorous mathematical proof of the convergence of a numerical method used to solve Equation (3).

In Section 2, we will re-formulate the problem and put it into perspective. In Section 3, we will investigate the properties of an operator K that arises from the re-formulated problem. The numerical method used to solve the problem will be defined in Section 4, and some preliminary properties of the associated numerical integral operators K_n will be explored. Several convergence theorems for the numerical integral operators K_n will be proven in Section 5. In Section 6, a convergence theorem for the numerical solution of the complete problem will be given. Finally, we will conclude with some closing remarks in Section 7.

2 Statement of the Problem

We begin with the integral equation in (4), namely

$$\lambda(t)\,\phi(t) - \gamma_1 \int_a^b |t-s|^{-1}[\phi(s) - \phi(t)]\,ds = \chi_1(t) \tag{5}$$

defined on an interval (a, b). Here we assume the constant $\gamma_1 > 0$ and the function $\chi_1 \in C([a,b])$. Furthermore, we assume the function λ is positive and continuous on (a, b) and that $\lambda(t) \to \infty$ as $t \to a$ and as $t \to b$. Under these assumptions, Equation (5) can be transformed to an integral equation of the second kind:

$$\phi(t) - \gamma(t) \int_a^b |t-s|^{-1}[\phi(s) - \phi(t)]\,ds = \chi(t) \tag{6}$$

defined on the interval [a, b]. Here $\gamma(t) := \gamma_1/\lambda(t) > 0$ on (a, b) and together with $\chi(t) := \chi_1(t)/\lambda(t)$ may be assumed to be continuous on $[a,b]$, because of the assumptions on λ. Define the operator

$$K\phi(t) := \int_a^b |t-s|^{-1}[\phi(s) - \phi(t)]\,ds \tag{7}$$

Then Equation (6) can be written in the familiar operator notation as

$$(I - \gamma K)\phi = \chi \tag{8}$$

Our problem is then to analyze the numerical solution of this equation when, in particular, a simple Nyström method (to be described in Section 4 below) is used.

The problem being addressed here differs from conventional weakly singular integral equations in at least two fundamental ways. First, while Equation (6) contains the difference term used in the well-known Singularity Subtraction Method, namely

$$K\phi(t) = \int_a^b k(t,s)\,[\phi(s) - \phi(t)]\,ds,$$

the subtracted term

$$\int_a^b k(t,s)\,\phi(t)\,ds = \phi(t)\int_a^b k(t,s)\,ds$$

in our case is divergent. This is in stark contrast to the conventional case where the subtracted term is and must be finite.

Second, for weakly singular integral of the second kind

$$\phi(t) - \int_a^b k_w(t,s)\phi(s)\,ds = \chi(t)$$

or

$$(I - K_w)\,\phi = \chi$$

where $|k_w(t,s)| \leq C\,|s-t|^{\alpha-1}, 0 < \alpha \leq 1$, K_w is compact from C([a,b]) $\to$ C([a,b]). Consequently, the analysis of a typical numerical method taking the form

$$(I - K_n)\phi_n = \chi \tag{9}$$

can be based on Anselone's Collectively Compact Operators, wherein the operators K_n are each compact from C([a,b]) $\to$ C([a,b]). (See, for example, [1]). Unfortunately, in our problem the operators are not compact, as we shall see below.

3 Mapping Properties of K

We first investigate the mapping properties of the operator in Equation (7):

$$K\phi(t) := \int_a^b |t-s|^{-1}\,[\phi(s) - \phi(t)]\,ds$$

We recall a function f is **uniformly Hölder continuous of order** α $(0 < \alpha \leq 1)$ on an interval $[a,b]$ if there exists a constant C such that

$$|\, f(x) - f(y) \,| \leq C\ |\, x - y \,|^{\alpha}$$

for all x and y in $[a, b]$. Define

$$C^{(0,\alpha)}([a,b]) := \begin{cases} \text{The space of all uniformly Hölder} \\ \text{continuous functions of order } \alpha \\ \text{on an interval } [a,b] \end{cases}$$

The following properties of $C^{(0,\alpha)}([a,b])$ are well-known:

Proposition 2. For $0 < \alpha < \beta \leq 1$

1. $C^{(0,\beta)}([a,b]) \subset C^{(0,\alpha)}([a,b])$
2. $C^{(0,\beta)}([a,b])$ is a subalgebra of $C([a,b])$
3. $C^{(0,\alpha)}([a,b])$ is a Banach space under the norm

$$\|f\|_\alpha = \|f\|_\infty + |f|_\alpha$$

where

$$|f|_\alpha := \sup\{\frac{|f(x) - f(y)|}{|x - y|^\alpha} \mid x \neq y\} \tag{10}$$

is a semi-norm.

4. Imbedding maps $I_{\beta,\alpha} : C^{(0,\beta)}([a,b]) \to C^{(0,\alpha)}([a,b])$ are compact.

Proof. (See [5, 3]) $\sqcup$

Corollary 3. For $0 < \alpha < \beta \leq 1$, the mapping $I^{-1}_{\beta,\alpha}$ from $(C^{(0,\beta)}([a,b]), \|\cdot\|_\alpha)$ onto $(C^{(0,\beta)}([a,b]), \|\cdot\|_\beta)$ is unbounded.

Proof. Otherwise, the identity map $I_{\beta,\beta}$ would be compact on the infinite dimension space $C^{(0,\beta)}([a,b])$. $\sqcup$

Corollary 4. For $0 < \alpha < \beta \leq 1$, $C^{(0,\beta)}([a,b])$ is not a Banach subspace of $C^{(0,\alpha)}([a,b])$.

Proof. Else $I^{-1}_{\beta,\alpha}$ would be bounded, by the closed graph theorem [4], since $I_{\beta,\alpha}$ and therefore $I^{-1}_{\beta,\alpha}$ are closed operators. $\sqcup$

Proposition 5. For $0 < \alpha < \beta \leq 1$, $K : C^{(0,\beta)}([a,b]) \to C^{(0,\alpha)}([a,b])$ is compact.

Proof. Mimicking the steps in one of the proofs in [3], one can show that K is bounded from $C^{(0,\beta)}([a,b]) \to C^{(0,\delta)}([a,b])$, where $\delta := (\alpha+\beta)/2$. Using the fact that the imbedding from $C^{(0,\delta)}([a,b]) \to C^{(0,\alpha)}([a,b])$ is compact, the proposition follows immediately. ⊔

Unfortunately, classical Fredholm theory does not apply here, because of the following observation.

Proposition 6. $C^{(0,\alpha)}([a,b])$ is not invariant under K for any $0 < \alpha \leq 1$.

Proof. By direct calculation, one can show that $\phi(t) := t^\beta \in C^{(0,\beta)}([a,b])$, but $K\phi(t) \notin C^{(0,\beta)}([a,b])$, assuming, without loss of generality, $[a,b] = [0,1]$. ⊔

For theoretical as well as numerical reasons, it is desirable to consider operators L whose range is contained in its domain, so that L^2, for example, is defined. This leads us to the following spaces. For $0 \leq \alpha < 1$, we define

$$X_\alpha := \bigcup \{ C^{(0,\beta)}([a,b]) \mid \alpha < \beta \leq 1 \}$$

In particular, X_0 is the set of all functions defined on [a, b] which are uniformly Hölder continuous of some order $\alpha \in (0,1]$.

Lemma 7. For $0 < \alpha < 1$, $(X_\alpha, \|\cdot\|_\alpha)$ is a normed linear space and is invariant under K.

Proof. $(X_\alpha, \|\cdot\|_\alpha)$ is a linear subspace of $C^{(0,\alpha)}([a,b]), \|\cdot\|_\alpha)$. The invariance follows from Proposition 4. ⊔

While the semi-norm $|\cdot|_\alpha$ in Equation (10) and hence the norm $\|\cdot\|_\alpha$ are defined for $\alpha \in (0,1]$ on $C^{(0,\alpha)}([a,b])$, it is convenient (and also consistent) to define

$$\|f\|_0 := \|f\|_\infty, \quad f \in X_0$$

Lemma 8. $(X_0, \|\cdot\|_0)$ is a normed linear space and is invariant under K.

Proof. $(X_0, \|\cdot\|_0) = (X_0, \|\cdot\|_\infty)$ is a linear subspace of $(C([a,b]), \|\cdot\|_\infty)$, and the invariance follows again from Proposition 4. ⊔

While not germane to our discussion here, it can be shown that the closure of $(X_\alpha, \|\cdot\|_\alpha)$ in $C^{(0,\alpha)}([a,b])$ is not $C^{(0,\alpha)}([a,b])$, even though $C^{(0,\alpha)}([a,b])$ contains $C^{(0,\beta)}([a,b])$ for all $\beta > \alpha$.

Proposition 9. K is unbounded on $(X_\alpha, \|\cdot\|_\alpha)$ for any $0 \le \alpha < 1$.

Proof. Assume, without loss of generality, $[a,b] = [0,1]$. One can then readily show that

$$\phi_n(t) := t^{\alpha+1/n}$$

is a bounded sequence in X_α, but $\|K\phi_n\|_\alpha \to \infty$ as $n \to \infty$. $\sqcup$

4 Numerical Integral Operators, K_n

The numerical solution of Equation (8) can be defined in terms of the following numerical integral operators. It is basically the operators associated with the Nyström method. For each integer $n > 0$, we define a partition P_n on the interval $[a,b]$ by partitioning the interval into n equal subintervals. We associate with the partition P_n the operator K_n defined on $C([a,b])$ as follows.

$$K_n\phi(t) := \sum_{j=1}^{k_n} w_{n,j}\ g_{n,j}(t)\ \Delta_{n,j}\phi(t), \qquad \phi \in C([a,b])$$

where

$$\begin{aligned}
k_n &= 2^n \\
w_{n,j} &= (b-a)/k_n =: h_n \\
g_{n,j}(t) &= \begin{cases} |t - t^*_{n,j}|^{-1} & t \notin [t_{n,j-1}, t_{n,j}] \\ \\ 2/h_n & t \in [t_{n,j-1}, t_{n,j}] \end{cases} \\
\Delta_{n,j}\phi(t) &= \phi(t^*_{n,j}) - \phi(t) \\
t_{n,j} &= a + jh_n, \quad j = 0, \ldots, k_n \\
t^*_{n,j} &= (t_{n,j-1} + t_{n,j})/2, \quad j = 1, \ldots, k_n
\end{aligned}$$

For simplicity, we have purposely chosen each weight $w_{n,j}$ associated with the j-th subinterval in P_n to be dependent only on the integer n and not on j. More sophisticated choice for the weights is of course possible, but the resulting analysis would be more complicated. Also $K_n\phi$ is actually defined for any function ϕ that is merely defined on the interval $[a,b]$. However, here we are only interested in those functions that are at least continuous. It is obvious that if $\phi \in C([a,b])$, then so is $K_n\phi$. That is, $C([a,b])$ is invariant under K_n.

The numerical solution ϕ_n to Equation (8) is now obtained by solving the equation

$$(I - \gamma K_n)\phi_n = \chi \tag{11}$$

By collocation at the k_n mid points $\{t^*_{n,j}\}$, the following system of linear equations is obtained

$$(I - \gamma K_n)\phi_n(t^*_{n,j}) = \chi(t^*_{n,j}), \quad j = 1, \ldots, k_n \tag{12}$$

from which $\{\phi_n(t^*_{n,j})\}$ can be solved.

For comparison with the operator K, we will look at some properties of K_n. Unless specified otherwise, we will always assume n is a positive integer in the following. Also for later convenience we introduce the following functions, each of which depends only on n:

$$\psi_n(t) := \sum_{j=1}^{k_n} w_{n,j}\, g_{n,j}(t)$$

We will first look at some properties of K_n on C([a,b]) and then its properties on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$. As we have already noted, we have

Proposition 10. C([a,b]) is invariant under K_n.

Moreover, we have

Proposition 11. K_n is bounded on C([a,b]) with $\|K_n\|_\infty = 8 \sum_{j=1}^{k_{n-1}} \frac{1}{2j-1}$

Proof. From the definition of K_n, it follows immediately that

$$K_n\phi(t) = K_{n,1}\phi(t) + K_{n,2}\phi(t)$$

where

$$\begin{aligned} K_{n,1}\phi(t) &= \sum_{j=1}^{k_n} w_{n,j}\ g_{n,j}(t)\ \phi(t^*_{n,j}), \\ K_{n,2}\phi(t) &= -\psi_n(t)\,\phi(t) \end{aligned}$$

Now $K_{n,1}$ is compact and hence bounded on C([a,b]), because it has finite dimensional range. Since $g_{n,j}(t) \in$ C([a,b]), so does $\psi_n(t)$. Hence $K_{n,2}$ is also bounded on C([a,b]). It follows that K_n must be bounded on C([a,b]). To find the norm of K_n on C([a,b]), let $t^* = (a+b)/2$. One can verify directly that

$$\|\psi_n\|_\infty = \psi_n(t^*)$$

For any $\phi \in$ C([a,b]),

$$\begin{aligned} |K_n\phi(t)| &\leq 2\|\phi\|_\infty \psi_n(t) \\ &\leq 2\|\phi\|_\infty \psi_n(t^*) \end{aligned}$$

Hence $\|K_n\|_\infty \leq 2\psi_n(t^*)$. Since $t^* \neq t^*_{n,j}$, $j = 1, \ldots, k_n$, there exists $\phi_o \in$ C([a,b]) such that $\|\phi_o\|_\infty = 1$ and $\phi_o(t^*) = -\phi_o(t^*_{n,j}) = 1$, $j = 1, \ldots, k_n$. Thus, $K_n\phi_o(t^*) = 2\psi_n(t^*)$ and $\|K_n\|_\infty \geq 2\psi_n(t^*)$. It follows that

$$\|K_n\|_\infty = 2\psi_n(t^*).$$

By direct verification, one obtains $\psi_n(t^*) = 4\sum_{j=1}^{k_{n-1}} \frac{1}{2j-1}$. Hence, $\|K_n\|_\infty = 8\sum_{j=1}^{k_{n-1}} \frac{1}{2j-1}$. ⊔

<u>Corollary 12.</u> K_n is not a compact operator on C([a,b]).

<u>Proof.</u> If K_n were compact, then $K_{n,2} = K_n - K_{n,1}$ would also be compact, since $K_{n,1}$ is compact. Now $\frac{1}{\psi_n(t)} \in$ C([a,b]), as $\psi_n(t)$ is bounded away from 0. Hence $\frac{K_{n,2}}{\psi_n(t)} = -$I would be compact. This is impossible, since the identity operator I is not compact on C([a,b]). ⊔

Incidentally, the last proposition implies that if the method of successive approximation is applied to Equation (8), it will likely fail as n increases, since $\|K_n\|_\infty$ are not uniformly bounded.

We now turn our attention to the properties of K_n on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$. Unlike the operator K, we have

<u>Proposition 13.</u> $C^{(0,\alpha)}([a,b])$ is invariant under K_n for $\alpha \in (0,1]$.

<u>Proof.</u> It suffices to show that $g_{n,j}(\cdot)\Delta_{n,j}\phi(\cdot) \in C^{(0,\alpha)}([a,b])$ for any $\phi \in C^{(0,\alpha)}([a,b])$. If $\phi \in C^{(0,\alpha)}([a,b])$, then clearly $\Delta_{n,j}\phi \in C^{(0,\alpha)}([a,b])$. One can also show that $g_{n,j} \in C^{(0,1)}([a,b])$ and hence it belongs to $C^{(0,\alpha)}([a,b])$ for $\alpha \in (0,1]$. Finally, $g_{n,j}(\cdot)\,\Delta_{n,j}\phi(\cdot) \in C^{(0,\alpha)}([a,b])$, since the latter is an algebra. ⊔

To investigate the boundedness of K_n on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$, it suffices to consider the individual components of K_n, leading us to define the following operators on $C^{(0,\alpha)}([a,b])$:

$$L_{n,j}\phi(t) := g_{n,j}(t)\,\Delta_{n,j}\phi(t), \quad j = 1, \ldots, k_n.$$

Clearly, $K_n = \sum_{j=1}^{k_n} w_{n,j}\, L_{n,j}$.

Lemma 14. $L_{n,j}$ is bounded on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$, for $j = 1,\ldots,k_n$ and $\|L_{n,j}\|_\alpha \leq 2\|g_{n,j}\|_\alpha$

Proof. For any $\phi \in C^{(0,\alpha)}([a,b])$, we have $|L_{n,j}\phi(t)| = |g_{n,j}(t)\,\Delta_{n,j}\phi(t)| \leq 2\|g_{n,j}\|_\infty \|\phi\|_\infty$, for all $t \in [a,b]$. Hence $\|L_{n,j}\phi\|_\infty \leq 2\|g_{n,j}\|_\infty \|\phi\|_\infty$. For any s and $t \in [a,b]$,

$$\begin{aligned} |L_{n,j}\phi(s) - L_{n,j}\phi(t)| &= |g_{n,j}(s)\,\Delta_{n,j}\phi(s) - g_{n,j}(t)\,\Delta_{n,j}\phi(t)| \\ &\leq \left| \left[g_{n,j}(s) - g_{n,j}(t)\right] \Delta_{n,j}\phi(s)\right| + |g_{n,j}(t)\left[\Delta_{n,j}\phi(s) - \Delta_{n,j}\phi(t)\right]| \\ &\leq 2\|\phi\|_\infty\, |g_{n,j}(s) - g_{n,j}(t)| + \|g_{n,j}\|_\infty\, |\phi(t) - \phi(s)| \\ &\leq (2\|\phi\|_\infty\, |g_{n,j}|_\alpha + \|g_{n,j}\|_\infty\, |\phi|_\alpha)\, |s-t|^\alpha \end{aligned}$$

since both $g_{n,j}$ and $\phi \in C^{(0,\alpha)}([a,b])$. Hence $|L_{n,j}\phi|_\alpha \leq 2\|\phi\|_\infty\, |g_{n,j}|_\alpha + \|g_{n,j}\|_\infty\, |\phi|_\alpha$. It follows that

$$\begin{aligned} \|L_{n,j}\phi\|_\alpha &= \|L_{n,j}\phi\|_\infty + |L_{n,j}\phi|_\alpha \\ &\leq 2\|g_{n,j}\|_\infty \|\phi\|_\infty + 2\|\phi\|_\infty\, |g_{n,j}|_\alpha + \|g_{n,j}\|_\infty\, |\phi|_\alpha \\ &\leq 2\|g_{n,j}\|_\alpha \|\phi\|_\alpha. \end{aligned}$$

The Lemma is now proved. ⊔

For $0 < \alpha < 1$, one can readily show that

$$\|g_{n,j}\|_\alpha = \|g_{n,j}\|_\infty + |g_{n,j}|_\alpha = \frac{2}{h_n} + \frac{\alpha^\alpha (1-\alpha)^{(1-\alpha)}}{h_n^{(1+\alpha)}}.$$

Proposition 15. K_n is bounded on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$.

Proof. This follows directly from the last Lemma. ⊔

While the restriction of the operator K to $C^{(0,\alpha)}([a,b])$ allows it to be defined, the restriction of the operator K_n to $C^{(0,\alpha)}([a,b])$ does not gain us much. We still have

Proposition 16. K_n is not compact on $C^{(0,\alpha)}([a,b])$ for $\alpha \in (0,1]$.

Proof. The proof follows in exactly the same manner as that in the C([a,b]) case. ⊔

Because K_n are not compact on $C^{(0,\alpha)}([a,b])$, we cannot make use of the theory of collectively compact operators to prove convergence of our numerical method. We do have

some type of compactness as we will see in the next proposition. However, this is mainly of academic interest only.

Proposition 17. $K_n : C^{(0,\alpha)}([a,b]) \to C^{(0,\beta)}([a,b])$ is compact, if $0 < \beta < \alpha \leq 1$.

Proof. If we denote by $A^{\alpha,\beta}$ the map $A : C^{(0,\alpha)}([a,b]) \to C^{(0,\beta)}([a,b])$, then $K_n^{\alpha,\beta} = I^{\alpha,\beta} K_n^{\alpha,\alpha}$. Since $K_n^{\alpha,\alpha}$ is bounded and $I^{\alpha,\beta}$ is compact, $K_n^{\alpha,\beta}$ is compact. ⊔

5 Convergence Theorems for K_n

As we cannot make use of the theory of collectively compact operators to prove the convergence of our numerical method, we resort to proving it directly. We will prove some pointwise convergence properties of K_n after establishing several preliminary lemmas. For convenience we define

$$\begin{aligned}
\Delta\phi(t,s) &:= \phi(t) - \phi(s) \\
g_\alpha(t,s) &:= g_{ss}(t,s)\,|t-s|^\alpha \\
B(t,\delta) &:= \{s \in [a,b] \mid \,|\, s-t \,|< \delta\} \\
F_\phi^\alpha(t,s) &:= \begin{cases} \frac{\phi(t)-\phi(s)}{|t-s|^\alpha} & \text{for } t \neq s \\ 0 & \text{for } t = s \end{cases}
\end{aligned}$$

where, as before,

$$g_{ss}(t,s) = \frac{1}{|t-s|}$$

Lemma 18. Let $\phi \in C^{(0,\beta)}([a,b])$, $\beta \in (0,1]$ and $\alpha \in (0,\beta]$. Then

$$|F_\phi^\alpha(t,s)| \leq \|\phi\|_\beta \, |t-s|^{\beta-\alpha}$$

for all $(t,s) \in [a,b] \times [a,b]$.

Proof. This follows trivially from the definition of $C^{(0,\beta)}([a,b])$.

Lemma 19. Let $t \in [a,b]$ and $\phi \in C^{(0,\beta)}([a,b])$, $\beta \in (0,1]$. Then for any $\epsilon > 0$, there exists $\delta(\epsilon,\phi) > 0$ independent of t such that

$$\left|\int_{B(t,\delta')} g_{ss}(t,s)\Delta\phi(s,t)\,ds\right| < \epsilon$$

for all $\delta' \le \delta$.

Proof. Let $t \in [x_1, x_2] \subset [a, b]$. Then

$$
\begin{aligned}
|\int_{x_1}^{x_2} g_{ss}(t, s)\Delta\phi(s, t)\, ds| &\le \int_{x_1}^{x_2} |g_\beta(t, s)F_\phi^\beta(t, s)|\, ds \\
\le \|\phi\|_\beta \int_{x_1}^{x_2} |t - s|^{\beta-1}\, ds &\le \left[\frac{2^{1-\beta}\|\phi\|_\beta}{\beta}\right](x_2 - x_1)^\beta
\end{aligned}
$$

Thus, the required δ can be chosen as

$$\delta = \frac{1}{2}\left(\frac{\beta\epsilon}{2^{1-\beta}\|\phi\|_\beta}\right)^{1/\beta}$$

$\sqcup$

Lemma 20. Let $t \in [a, b]$ and $\phi \in C^{(0,\beta)}([a, b])$, $\beta \in (0, 1]$. Then for any $\epsilon > 0$ there exist N (independent of t), $0 \le k_{N,1} < k_{N,2}$, and $\delta > 0$ such that

$$| S_n(\tau) := \sum_{j=k_{n,1}+1}^{k_{n,2}} w_{n,j}\; g_{n,j}(\tau)\; \Delta_{n,j}\phi(\tau) \;| < \epsilon$$

for all $\tau \in B(t, \delta)$ and for all $n \ge N$, where

$$
\begin{aligned}
a + k_{n,1}h_n &= a + k_{N,1}h_N =: a_1 \\
a + k_{n,2}h_n &= a + k_{N,2}h_N =: b_1 \quad \text{and} \\
B(t, \delta) &\subset [a_1, b_1]
\end{aligned}
$$

Proof. Let $\epsilon > 0$ be given. Let $N = \max\{N_1, N_2, N_3\}$, where N_1, N_2, N_3 are specified below. For any positive integer n, define

$$I_{n,j} := \begin{cases} \left[t_{n,j-1}, t_{n,j}\right) & \text{if } 1 \le j < k_n \\ \left[t_{n,j-1}, t_{n,j}\right] & \text{if } j = k_n \end{cases}$$

Since $[a, b]$ is the disjoint union of $\{I_{n,j}\}_{j=1}^{j=k_n}$, $t \in I_{n,j_n^*(t)}$ for a unique $j_n^*(t)$, $1 \le j_n^*(t) \le k_n$. We define $[a_1, b_1] := \left[t_{N,j_N^*(t)-1}, t_{N,j_N^*(t)}\right]$. It follows that $[a_1, b_1] \supseteq I_{n,j_n^*(t)}$ and $k_{n,1} < j_n^*(t) \le$

$k_{n,2}$ for all $n \geq N$. For ease of presentation, we assume $t \in (a_1, b_1)$. (If, for example, $t = t_{N,j_N^*-1}$ and $t \neq a$, we can increase N by 1, and let $[a_1, b_1] := \left[t_{N,j_N^*-2}, t_{N,j_N^*}\right]$.) In the following, we will always assume $\tau \in (a_1, b_1) = (k_{n,1}h_n, k_{n,2}h_n)$ and $n \geq N$. Then

$$S_n(\tau) = S_{n,1}(\tau) + S_{n,2}(\tau) + S_{n,3}(\tau)$$

where

$$\begin{aligned}
S_{n,1}(\tau) &= \sum_{k_{n,1} < j < j_n^*(\tau)} w_{n,j}\, g_{n,j}(\tau)\, \Delta_{n,j}\phi(\tau) \\
S_{n,2}(\tau) &= w_{n,j_n^*(\tau)}\, g_{n,j_n^*(\tau)}(\tau)\, \Delta_{n,j_n^*(\tau)}\phi(\tau) \\
S_{n,3}(\tau) &= \sum_{j_n^*(\tau) < j \leq k_{n,2}} w_{n,j}\, g_{n,j}(\tau)\, \Delta_{n,j}\phi(\tau)
\end{aligned}$$

We shall show that each of the above can be made arbitrarily small uniformly for $\tau \in (a_1, b_1)$ for n sufficiently large. Indeed,

$$\begin{aligned}
|\, S_{n,2}(\tau)\,| &= 2|\phi(t^*_{n,j_n^*(\tau)}) - \phi(\tau)\,| \\
&\leq 2\,\|\phi\|_\beta\, |t^*_{n,j_n^*(\tau)} - \tau|^\beta \\
&\leq 2\,\|\phi\|_\beta\, h_N^\beta \\
&\leq \frac{\epsilon}{3}
\end{aligned}$$

provided

$$n \geq N_2 := \max\left\{\lceil \frac{1}{\beta}\log_2\left(\frac{6}{\epsilon}(b-a)^\beta\, \|\phi\|_\beta\right)\rceil, 1\right\}$$

We will now derive a bound for $|\, S_{n,1}(\tau)\,|$. If $j_n^*(\tau) = k_{n,1} + 1$, then $S_{n,1}(\tau) = 0$. Otherwise, if $j_n^*(\tau) > k_{n,1} + 1$, then

$$\begin{aligned}
|\, S_{n,1}(\tau)\,| &\leq h_n \sum_{k_{n,1} < j < j_n^*(\tau)} \frac{|\,\phi(t^*_{n,j}) - \phi(\tau)\,|}{|\,t^*_{n,j} - \tau\,|} \\
&\leq h_n\, \|\phi\|_\beta \sum_{k_{n,1} < j < j_n^*(\tau)} (\tau - t^*_{n,j})^{\beta-1} \\
&= h_n\, \|\phi\|_\beta \sum_{m=0}^{L_n} (x + m h_n)^{\beta-1}
\end{aligned}$$

where $x := \tau - t^{*}_{n,j^{*}_{n}(\tau)-1}$ and $L_n := j^{*}_{n}(\tau) - k_{n,1} - 2$. From the definitions of $t^{*}_{n,j}$ and $j^{*}_{n}(\tau)$, we must have $\frac{h_n}{2} \leq x \leq \frac{3h_n}{2}$. Let $y := \frac{x}{h_n}$. Then $\frac{1}{2} \leq y \leq \frac{3}{2}$ and

$$
\begin{aligned}
|\, S_{n,1}(\tau) \,| &\leq h_n^{\beta} \, \|\phi\|_{\beta} \sum_{m=0}^{L_n} (y+m)^{\beta-1} \\
&= h_n^{\beta} \, \|\phi\|_{\beta} \, y^{\beta-1} + h_n^{\beta} \, \|\phi\|_{\beta} \sum_{m=1}^{L_n} (y+m)^{\beta-1}
\end{aligned}
$$

Since $\beta \in (0,1]$ and $\frac{1}{2} \leq y \leq \frac{3}{2}$,

$$
\begin{aligned}
|\, S_{n,1}(\tau) \,| &\leq 2^{1-\beta} h_n^{\beta} \, \|\phi\|_{\beta} + h_n^{\beta} \, \|\phi\|_{\beta} \sum_{m=1}^{L_n} m^{\beta-1} \\
&\leq 2^{1-\beta} h_n^{\beta} \, \|\phi\|_{\beta} + h_n^{\beta} \, \|\phi\|_{\beta} \, (1 + \frac{1}{\beta} L_n^{\beta})
\end{aligned}
$$

From $j^{*}_{n}(\tau) \leq k_{n,2}$ and $L_n = j^{*}_{n}(\tau) - k_{n,1} - 2$ we have

$$
\begin{aligned}
L_n h_n &< (k_{n,2} - k_{n,1}) \, h_n \\
&= b_1 - a_1
\end{aligned}
$$

and so

$$
\begin{aligned}
|\, S_{n,1}(\tau) \,| &\leq (2^{1-\beta} + 1) \, h_n^{\beta} \, \|\phi\|_{\beta} + \frac{1}{\beta} \|\phi\|_{\beta} \, (b_1 - a_1)^{\beta} \\
&\leq \frac{\epsilon}{3}
\end{aligned}
$$

provided

$$
n \geq N_1 := \max\{N_{1,a}, N_{1,b}\}
$$

where

$$
\begin{aligned}
N_{1,a} &= \max\{ \lceil \frac{1}{\beta} \log_2 \left(\frac{18}{\epsilon} (b-a)^{\beta} \|\phi\|_{\beta} \right) \rceil, 1 \} \\
N_{1,b} &= \max\{ \lceil \frac{1}{\beta} \log_2 \left(\frac{6}{\epsilon\beta} (b-a)^{\beta} \|\phi\|_{\beta} \right) \rceil, 1 \}
\end{aligned}
$$

Estimating a bound for $|\, S_{n,3}(\tau) \,|$ is similar to that for $|\, S_{n,1}(\tau) \,|$, producing N_3 similar to N_1. The Lemma is proved by choosing $\delta = \min\{|t - a_1|, |t - b_1|\}$. ⊔

Remark. It follows from the proof of the last lemma that

$$
|\sum_{j=m_1}^{m_2} w_{n,j} \; g_{n,j}(\tau) \; \Delta_{n,j} \phi(\tau) \,| < \epsilon
$$

as long as $k_{n,1} + 1 \le m_1 \le m_2 \le k_{n,2}$.

Proposition 21. For any $\phi \in C^{(0,\beta)}([a,b])$, $\beta \in (0,1]$,

$$\lim_{n\to\infty} \|(K - K_n)\phi\|_\infty = 0$$

Proof. Let $\epsilon > 0$ be given. By the compactness of [a,b], it suffices to show that for any t in [a,b], there exist N_t and δ_t such that $|(K - K_n)\phi(\tau)| < \epsilon$ for all $n > N_t$ and for all $\tau \in B(t, \delta_t)$. Now for any γ_1 and γ_2 with $0 \le \gamma_1 \le t - a$, and $0 \le \gamma_2 \le b - t$, and for any m_1 and m_2 with $1 \le m_1 < m_2 \le k_n$,

$$\begin{aligned}(K - K_n)\phi(\tau) &= T_1(\tau, a, t-\gamma_1) + T_1(\tau, t-\gamma_1, t+\gamma_2) + T_1(\tau, t+\gamma_2, b)\\ &- T_2(\tau, n, 1, m_1) - T_2(\tau, n, m_1+1, m_2) - T_2(\tau, n, m_2+1, k_n)\end{aligned}$$

where

$$\begin{aligned}T_1(\tau, x, y) &= \int_x^y g_{ss}(t,s)\Delta\phi(s,t)\, ds\\ T_2(\tau, n, m, k) &= \sum_{j=m}^{k} w_{n,j}\; g_{n,j}(\tau)\; \Delta_{n,j}\phi(\tau)\end{aligned}$$

It follows readily from the last two lemmas that there exist N_1 and $\delta_1 > 0$ such that by letting $\gamma_1 = t - t_{N_1, k_{N_1},1} \ge 0$ and $\gamma_2 = t_{N_1, k_{N_1},2} - t \ge 0$, we have

$$\begin{aligned}|T_1(\tau, t-\gamma_1, t+\gamma_2)| &\le \frac{\epsilon}{3}\\ |T_2(\tau, n, k_{n,1}+1, k_{n,2})| &\le \frac{\epsilon}{3}\end{aligned}$$

for all $n \ge N_1$ and $\tau \in B(t, \delta_1)$. Let $f(t,s) := g_{ss}(t,s)\Delta\phi(s,t)$. Since $f(t,s)$ is continuous and therefore uniformly continuous on $[[t-\delta_1, t+\delta_1] \cap [a,b]] \times [a, t-\gamma_1]$, there exist N_2 and $\delta_2 > 0$ such that

$$\begin{aligned}|T_1(\tau, a, t-\gamma_1) &- T_2(\tau, n, 1, k_{n,1})|\\ &= |\sum_{j=1}^{k_{n,1}} \int_{t_{n,j-1}}^{t_{n,j}} [f(\tau,s) - f(\tau, t^{*}_{n,j})]\, ds|\\ &\le \frac{\epsilon}{6}\end{aligned}$$

for all $n \geq N_2$ and $\tau \in B(t, \delta_2)$. Similarly, there exist N_3 and $\delta_3 > 0$ such that

$$\begin{aligned} |T_1(\tau, t + \gamma_2, b) &- T_2(\tau, n, k_{n,2} + 1, k_n)| \\ &= |\sum_{j=k_{n,2}+1}^{k_n} \int_{t_{n,j-1}}^{t_{n,j}} [f(\tau, s) - f(\tau, t^{*}_{n,j})] \, ds| \\ &\leq \frac{\epsilon}{6} \end{aligned}$$

for all $n \geq N_3$ and $\tau \in B(t, \delta_3)$. The proposition is proved by letting $N = \max\{N_1, N_2, N_3\}$ and $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. $\sqcup$

If follows immediately from the definition of X_0 that

<u>Corollary 22.</u> For any $\phi \in X_0$,

$$\lim_{n \to \infty} \|(K - K_n)\phi\|_\infty = 0$$

If $\phi \in C^{(0,\beta)}([a, b])$, $\beta \in (0, 1]$, then we already know that both $K\phi$ and $K_n\phi \in C^{(0,\alpha)}([a, b])$, for $\alpha \in (0, \beta)$. One may suspect the convergence in the last proposition is also true in $C^{(0,\alpha)}([a, b])$, i.e., $\lim_{n \to \infty} \|(K - K_n)\phi\|_\alpha = 0$ for $0 < \alpha < \beta$. Unfortunately, we have not been able to prove it. However, for those $\phi \in C^{(0,\beta)}([a, b])$ that satisfy the additional assumption in the following proposition, we do have convergence in certain $C^{(0,\alpha)}([a, b])$.

<u>Proposition 23.</u> Let $\phi \in C^{(0,\beta)}([a, b])$, $\beta \in (0, 1]$ such that

$$|K_n\phi(t) - K_n\phi(s)| \leq C_\phi |t - s|^\xi$$

for some $0 < \xi < \beta$ and for some constant C_ϕ **independent** of n, then

$$\lim_{n \to \infty} \|(K - K_n)\phi\|_\alpha = 0$$

for $0 < \alpha < \xi$.

<u>Proof.</u> Because of the last proposition, it suffices to prove $\lim_{n \to \infty} |(K - K_n)\phi|_\alpha = 0$. For simplicity of notation, let

$$\Delta_n K\phi(t, s) := (K - K_n)\phi(t) - (K - K_n)\phi(s)$$

Then for $0 < \gamma < 1$,

$$\begin{aligned} |\Delta_n K\phi(t,s)| &= |\Delta_n K\phi(t,s)|^{\gamma} \, |\Delta_n K\phi(t,s)|^{1-\gamma} \\ &= |K\phi(t) - K\phi(s) - K_n\phi(t) + K_n\phi(s)|^{\gamma} \, |\Delta_n K\phi(t,s)|^{1-\gamma} \\ &\leq (|K\phi(t) - K\phi(s)| + |K_n\phi(t) - K_n\phi(s)|)^{\gamma} \, |\Delta_n K\phi(t,s)|^{1-\gamma} \end{aligned}$$

Since $\phi \in C^{(0,\beta)}([a,b])$, $K\phi \in C^{(0,\xi)}([a,b])$, as $\xi < \beta$. Hence

$$\begin{aligned} |\Delta_n K\phi(t,s)| &\leq (|K\phi|_{\xi} + C_{\phi})^{\gamma} |t-s|^{\xi\gamma} \, |\Delta_n K\phi(t,s)|^{1-\gamma} \\ &\leq C_1 |t-s|^{\xi\gamma} \|K\phi - K_n\phi\|_{\infty}^{1-\gamma} \end{aligned}$$

where $C_1 = 2^{1-\gamma}(|K\phi|_{\xi} + C_{\phi})^{\gamma}$. The proposition follows by letting $\gamma = \frac{\alpha}{\xi}$. ⊔.

6 Convergence Theorems for $I - \gamma K_n$

In studying the properties of the operator $A_n := I - \gamma K_n$, the matrix B_n defined below will be useful. For any $\gamma \in C([a,b])$ with $\gamma(t) > 0$ in (a,b), let

$$c_{n,j}(t) := \gamma(t)\, w_{n,j}\, g_{n,j}(t) > 0 \text{ on (a,b)}, \quad j = 1, \ldots, k_n$$

and

$$b_n(t) := 1 + \sum_{j=1}^{k_n} c_{n,j}(t) > 0 \text{ on [a,b]}$$

Then

$$B_n := (b_{i,j})$$

where

$$b_{i,j} = b_n(t^*_{n,i})\, \delta_{i,j} - c_{n,j}(t^*_{n,i}), \quad 0 \leq i,j \leq k_n$$

B_n is simply the discretized version of A_n in the sense that

$$B_n(\phi(t^*_{n,i}))_{i=1,\ldots,k_n} = (A_n\phi(t^*_{n,i}))_{i=1,\ldots,k_n}$$

Lemma 24. B_n is invertible.

Proof. Let $\Lambda_i := \sum_{j\neq i} c_{n,j}(t^*_{n,i})$, $i = 1, \ldots, k_n$. One can directly verify that $b_{i,i} = 1 + \Lambda_i$, $i = 1, \ldots, k_n$ Thus, applying Gerschgorin Circle Theorem [8], all eigenvalues of B_n are contained in the union of the disks $|z - b_{i,i}| \leq \Lambda_i$, $1 \leq i \leq k_n$. It follows that all the eigenvalues must have absolute values ≥ 1. Hence, B_n is invertible. ⊔

Proposition 25. Let $\gamma \in$ C([a,b]) and $\gamma(t) > 0$ on (a, b). Then for each positive integer n, A_n maps C(J) 1-1 onto C(J), where $J := [a, b]$.

Proof. A_n is clearly defined on C(J). For a given $\chi \in$ C(J), we can define, because of the invertibility of B_n,

$$\begin{aligned} \vec{\chi}_n &:= (\chi(t^*_{n,1}), \ldots, \chi(t^*_{n,N}))^t \\ (\phi^\chi_i) &:= B_n^{-1} \vec{\chi}_n \end{aligned}$$

Then it can readily be shown that $A_n \phi = \chi$, where

$$\phi(t) = b_n(t)^{-1} [\chi(t) + \sum_{j=1}^{k_n} c_{n,j}(t) \phi^\chi_j] \tag{13}$$

Hence A_n is onto. If $A_n \phi = 0$, then $A_n \phi(t^*_{n,i}) = 0$, $i = 1, \ldots, k_n$. Since B_n is invertible, $\phi(t^*_{n,i}) = 0$, $i = 1, \ldots, k_n$. Subsequently, $\phi(t) = 0$, since $b_n > 0$. ⊔

Corollary 26. A_n^{-1} is bounded for each n.

Proof. Clearly $A_n = I - \gamma K_n$ is bounded on the Banach space $C([a, b])$. The boundedness of A_n^{-1} is a consequence of the Open Mapping Theorem [2]. ⊔

Other properties of the matrix B_n that we will need are contained in the following lemmas.

Lemma 27. B_n is irreducible.

Proof. Because B_n is a full matrix with no non-zero entries, it is irreducible [8].

Lemma 28. $B_n^{-1} > 0$, i.e., all entries are positive.

Proof. B_n is real, irreducible, diagonally dominant with

$$b_{i,j} = \begin{cases} < 0, & i \neq j \\ > 0, & i = j \end{cases} \tag{14}$$

The Lemma now follows from a theorem in Varga [8] (p. 85). ⊔

Lemma 29. Each row-sum of B_n is 1.

Proof. The i-th row-sum of B_n is $\sum_{j=1}^{N} b_{i,j}$. From the definition of $b_{i,j}$, each is seen to be one.

Remarks. Since the entries $b_{i,j}$ of B_n do not all have the same signs,

$$\|B_n\|_\infty = \max_{1\le i\le k_n} \sum_{1=j}^{k_n} |b_{i,j}| \neq 1.$$

Lemma 30. If each row-sum of a non-singular $n\times n$ matrix $A=(a_{i,j})$ is 1, then its inverse has the same property.

Proof. Let $A_j, j=1,\ldots,n$, denote the $n\times n$ matrix which is identical to A except for the j-th column where it consists of all ones. Because

$$\sum_{j=1}^{n} a_{i,j} = 1,\ i=1,\ldots,n$$

the determinant of A is the same as the determinant of A_j for $1\le j\le n$. The i-th row sum of A^{-1} is the sum of the cofactors along the i-th column of A divided by the determinant of A. However, the sum of the cofactors along the i-th column of A is just the determinant of A_i, which is also the determinant of A. Hence the i-th row sum of A^{-1} is one. ⊔

Corollary 31. $\|B_n^{-1}\|_\infty = 1$ for all positive integer n.

Proof. This follows from the last lemma and the fact that all entries in B_n^{-1} is positive.

Proposition 32. $\left(A_n^{-1}\right)_{n=1}^{\infty}$ is uniformly bounded on C([a,b]).

Proof. Let $\chi\in$ C([a,b]). Then using the notations in Equation (13), we have

$$A_n^{-1}\chi(t) = b_n(t)^{-1}\left[\chi(t)+\vec{C}_n(t)B_n^{-1}\vec{\chi}_n\right]$$

where

$$\vec{C}_n(t) := (c_{n,1}(t),\ldots,c_{n,N}(t)),\ N=k_n$$

From the definition of $b_n(t)$, it follows immediately that

$$\begin{aligned} |b_n(t)^{-1}| &< 1 \quad \text{and} \\ \|b_n(t)^{-1}\vec{C}_n(t)\|_1 &\le 1 \end{aligned}$$

for all $t \in [a,b]$ and $n > 0$. Also,

$$\begin{aligned} \|B_n^{-1}\vec{\chi}_n\|_\infty &\leq \|B_n^{-1}\|_\infty \|\vec{\chi}_n\|_\infty \\ &= \|\vec{\chi}_n\|_\infty \\ &\leq \|\chi\|_\infty \end{aligned}$$

It readily follows that

$$\|A_n^{-1}\chi\|_\infty \leq 2\,\|\chi\|_\infty$$

for each χ in C([a,b]) and for all $n > 0$. Hence $\|A_n^{-1}\|_\infty \leq 2$ for all n. $\sqcup$

<u>Theorem 33.</u> Let $\chi \in C([a,b])$ and assume $(I - \gamma K)\phi = \chi$ has a unique solution $\phi \in X_0$. For each positive integer n, let ϕ_n be the solution of

$$A_n\phi_n := (I - \gamma K_n)\phi_n = \chi$$

Then

$$\|\phi - \phi_n\|_\infty \to 0$$

as $n \longrightarrow \infty$.

<u>Proof.</u> Following the standard arguments, we have

$$\begin{aligned} 0 &= (I-\gamma K)\phi - (I - \gamma K_n)\phi_n \\ &= \phi - \phi_n - \gamma(K\phi - K_n\phi_n) \\ &= \phi - \phi_n - \gamma(K\phi - K_n\phi + K_n\phi - K_n\phi_n) \\ &= \phi - \phi_n - \gamma\left((K - K_n)\phi + K_n(\phi - \phi_n)\right) \\ &= (I - \gamma K_n)(\phi - \phi_n) - \gamma(K - K_n)\phi \\ &= A_n(\phi - \phi_n) - \gamma(K - K_n)\phi \end{aligned}$$

Hence

$$\phi - \phi_n = \gamma A_n^{-1}(K - K_n)\phi.$$

and

$$\|\phi - \phi_n\|_\infty \leq C\|A_n^{-1}\|_\infty \|(K - K_n)\phi\|_\infty.$$

The theorem follows from the uniform boundedness of A_n^{-1} and the pointwise convergence of K_n to K. $\sqcup$

7 Conclusion

In this report we have analyzed the numerical solution of the singular integral equation

$$(I - \gamma K)\phi = \chi \tag{15}$$

using the Nyström method described in Section 4. Here

$$K\phi(t) = \int_a^b |t-s|^{-1} \, [\phi(s) - \phi(t)] \, ds$$

We studied the mapping properties of the operator K and found that the space $(X_0, \|\cdot\|_\infty)$ of all uniformly Hölder continuous functions, despite not being a Banach space, is a natural setting to study the unbounded operator K, as it (X_0) is invariant under K.

We also studied the mapping properties of the numerical integral operators K_n that arise from the Nyström method. It is found that K_n are bounded on C([a,b]) and (therefore) on X_0, but they are not compact on C([a,b]). Nevertheless, we proved a pointwise convergence theorem of K_n to K on $(X_0, \|\cdot\|_\infty)$. Using this and other properties of K_n, we proved, under appropriate conditions, the convergence of the numerical solutions of the singular integral equation (15) to its actual solution.

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